

IDEMPOTENT FUNCTORS THAT PRESERVE COFIBER SEQUENCES AND SPLIT SUSPENSIONS

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ABSTRACT. We show that an f -localization functor L_f commutes with cofiber sequences of $(N - 1)$ -connected finite complexes if and only if its restriction to the collection of $(N - 1)$ -connected finite complexes is R -localization for some unital subring $R \subseteq \mathbb{Q}$. This leads to a homotopy-theoretical characterization of the rationalization functor: the restriction of L_f to simply-connected spaces (not just the finite complexes) is rationalization if and only if $L_f(S^2)$ is nontrivial and simply-connected, L_f preserves cofiber sequences of simply-connected finite complexes, and for each simply-connected finite complex K , $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for large enough k and various values of n .

INTRODUCTION

Let $f : P \rightarrow Q$ be a map from one CW complex to another. The f -localization functor L_f is the universal example of a homotopically idempotent functor from spaces to spaces which carries f to a weak equivalence. These functors are the primary—conceivably the only¹—examples of homotopically idempotent functors.

The R -localization functors are among the earliest and best-behaved examples of localization functors, defined for (unital) subrings $R \subseteq \mathbb{Q}$; we denote them by L_R . There are various constructions of these, but all of them agree on simply-connected spaces. They were constructed with the intention of lifting to spaces the algebraic operation of R -localization applied to homotopy and homology groups. This prescribed algebraic effect guarantees two nice homotopy-theoretical properties of R -localization: L_R carries cofiber sequences of simply-connected spaces to cofiber sequences and similarly for fiber sequences.

Rationalization is the special case $R = \mathbb{Q}$, and here the theory is most powerful. The central theorems of Quillen [10] and Sullivan [13] show

2010 *Mathematics Subject Classification*. Primary 55P60, 55P62; Secondary 55P35, 55P40.

Key words and phrases. localization, rationalization, suspension.

¹It has been shown [5] that it is impossible to prove in ZFC that all homotopically idempotent functors take the form L_f for some f ; but in the presence of Vopěnka's Principle (which is thought, but not known, to be consistent with ZFC), every homotopically idempotent functor *is* of the form L_f for some well-chosen map f .

that the homotopy theory of simply-connected rational spaces (i.e., simply-connected spaces for which $X \rightarrow L_{\mathbb{Q}}(X)$ is a weak equivalence) is perfectly modeled by the algebraic homotopy theories of DGLAs and of CDGAs, respectively. Rational spaces have two additional homotopy-theoretic properties beyond those enjoyed by all simply-connected R -local spaces: their suspensions split as wedges of copies of $L_{\mathbb{Q}}(S^n)$ for various values of n ; and, dually, their loop spaces split as products of copies of $K(\mathbb{Q}, n)$.

Many noncomputational theorems about rationalizations which were first proved via the algebraic machinery provided by Quillen and Sullivan can also be proved by appealing to the basic homotopy-theoretical properties noted above. For example, the celebrated Mapping Theorem for Lusternik-Schnirelmann category was originally proved using Sullivan models [7], but later a much simpler proof was found that was based on the splitting of loop spaces [8]. Examples like this led the author to wonder if any localization functor L_f that satisfies the four homotopy-theoretical properties listed above must be related to rationalization in some way.

In fact, rationalization is determined—overdetermined!—by these properties: it is the only localization functor of simply-connected spaces that preserves cofiber sequences and splits suspensions. Furthermore, the restriction of R -localization to simply-connected finite complexes is characterized by the fact that it commutes with cofiber sequences. These are the main results of this paper.

We write $\mathcal{K}(N)$ for the collection of all $(N-1)$ -connected finite complexes.

Theorem 1. *Let $f : P \rightarrow Q$ be a map of CW complexes. Let $N \geq 2$ and suppose $L_f(S^N)$ is simply-connected and not contractible. Then the following are equivalent:*

- (1) *the restriction of L_f to $\mathcal{K}(N)$ is R -localization for some (unital) subring $R \subseteq \mathbb{Q}$,*
- (2) *L_f commutes with cofiber sequences² of $(N-1)$ -connected finite complexes.*

Taking $N = 2$ in Theorem 1, we obtain a characterization of the rationalization of simply-connected spaces in terms of the elementary notions of homotopy theory.

Theorem 2. *The restriction of a localization functor L_f to simply-connected spaces is rationalization if and only if the following three conditions hold:*

- *$L_f(S^2)$ is simply-connected and not weakly contractible,*
- *L_f commutes with cofiber sequences of simply-connected finite complexes, and*
- *if K is a simply-connected finite complex, then for some $k \in \mathbb{N}$, the suspension $\Sigma^k L_f(K)$ splits as a wedge of copies of $L_f(S^n)$ for various values of n .*

²For any f , there is a natural comparison map $\xi : C_{L_f(\alpha)} \rightarrow L_f(C_\alpha)$; L_f commutes with the cofiber sequence $X \xrightarrow{\alpha} Y \rightarrow C_\alpha$ if this transformation is a weak equivalence.

We conclude the introduction with two bits of speculation.

Our proof of Theorem 1 comes very close to showing that L_f commutes with cofiber sequences (and hence restricts to R -localization on $\mathcal{K}(N)$) if and only if L_f ‘respects the smash and suspension structure of spheres’—that is, if $\Sigma L_f(S^n)$ and $L_f(S^n) \wedge L_f(S^m)$ are f -local for all $m, n \geq N$. Using a relative version of the theory of resolving classes³, we have been able to prove this equivalence under the additional assumption that f factors up to homotopy through a finite-dimensional complex (we have not included that proof here). Is it true for all f ?

Second, let us say that L_f ‘respects connectivity’ if for any n -connected $K \in \mathcal{K}(N)$, $L_f(K)$ is also n -connected. If L_f respects connectivity, must it satisfy the conditions of Theorem 1?

Acknowledgement. The author gratefully acknowledges the assistance of Steven Landsburg (via MathOverflow) on the algebraic implications of the vanishing of Tor. Javier Gutiérrez provided valuable feedback on an earlier version of this paper, as did the referee.

1. PRELIMINARIES

We’ll work in a fixed convenient category \mathcal{T}_* of pointed topological spaces; for example \mathcal{T}_* could be the category of compactly generated weak Hausdorff spaces. If so inclined, the reader may pretend that this paper was written simplicially.

We write $\text{conn}(X) = n$ if X is n -connected but not $(n+1)$ -connected.

1.1. Localization Functors. Let $f : P \rightarrow Q$ be a map between pointed CW complexes. A pointed space X is said to be **f -local** if the induced map

$$f^* : \text{map}_*(Q, X) \longrightarrow \text{map}_*(P, X)$$

of pointed mapping spaces is a weak homotopy equivalence; a map $q : X \rightarrow Y$ is called an **f -equivalence** if for every f -local space Z the induced map

$$q^* : \text{map}_*(Y, Z) \longrightarrow \text{map}_*(X, Z)$$

is a weak homotopy equivalence. A map $i : X \rightarrow L$ is said to be an **f -localization** of X if L is f -local and i is an f -equivalence.

The following important properties follow easily from the definitions.

Lemma 3.

- (a) *An f -equivalence between f -local spaces is a weak equivalence.*
- (b) *Let $i : X \rightarrow L$ be f -localization and let $g : X \rightarrow Z$ with Z f -local. Then in the diagram*

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ i \downarrow & \nearrow \gamma & \\ L & & \end{array}$$

³See [12] for an account of the absolute theory.

there is a map γ , unique up to homotopy, making the triangle commute up to homotopy.

An **coaugmented functor** is a functor $F : \mathcal{T}_* \rightarrow \mathcal{T}_*$ equipped with a natural transformation (its coaugmentation) $\iota : \text{id} \rightarrow F$. A coaugmented functor F is **homotopically idempotent** if for every X , the maps

$$\iota_{F(X)}, F(\iota_X) : F(X) \longrightarrow F(F(X))$$

are homotopic to one another, and both are weak equivalences. An **f -localization functor** is a homotopically idempotent functor L_f such that, for every space X , the coaugmentation $\iota_X : X \rightarrow L_f(X)$ is an f -localization of X .

The main existence theorem is as follows [6, Thm.1.A.3].

Theorem 4 (Bousfield, Farjoun). *For any map $f : P \rightarrow Q$ between CW complexes, there exists an f -localization functor L_f .*

Here are some basic properties of f -localization.

Proposition 5. *Let $f : P \rightarrow Q$ be a map of CW complexes.*

- (a) *If $q : X \rightarrow Y$ is an f -equivalence, then so is $q \wedge \text{id}_Z$ for any space Z .*
- (b) *$L_f(q)$ is a weak equivalence if and only if q is an f -equivalence.*
- (c) *For any X and Y , the natural map $L_f(X \times Y) \rightarrow L_f(X) \times L_f(Y)$ is a weak equivalence.*

Proof. Parts (a) and (b) are Example 1.D.5 and Proposition 1.C.5 in [6], respectively. Part (c) is [6, 1.A.8(e.4)]. \square

If $q : X \rightarrow Y$ is an f -equivalence, then every f -local space is also q -local, and so the transformation j_{L_f} in the square

$$\begin{array}{ccc} \text{id} & \xrightarrow{j} & L_q \\ \downarrow \iota & & \downarrow L_f(\iota) \\ L_f & \xrightarrow{j_{L_f}} & L_q \circ L_f \end{array}$$

evaluates to a weak equivalence $j_{L_f(X)} : L_f(X) \rightarrow L_q(L_f(X))$ for every space X . Thus we say that $\iota : \text{id} \rightarrow L_f$ factors through $j : \text{id} \rightarrow L_q$ ‘up to weak homotopy equivalence’. Since $L_q \circ L_f(X) \sim L_f(X)$ for all X , the functor $L_q \circ L_f$ is a perfectly good choice of f -localization functor. Thus we will abuse notation by silently redefining L_f and referring to $L_f(\iota)$ as a **comparison map** $\xi : L_q \rightarrow L_f$.

Proposition 6. *Suppose q is an f -equivalence. If $L_q(X)$ is f -local, then the comparison $\xi : L_q(X) \rightarrow L_f(X)$ is a weak equivalence.*

Proof. Any q -equivalence is a fortiori an f -equivalence. Since $X \rightarrow L_q(X)$ is a q -equivalence by definition and $L_q(X)$ is f -local by hypothesis, we have weak equivalences $L_f(X) \rightarrow L_f(L_q(X)) \leftarrow L_q(X)$. \square

We conclude with two useful results about the behavior of f -localization with respect to homotopy colimits and homotopy limits.

Proposition 7. *Let $f : P \rightarrow Q$ be a map of CW complexes, and write \mathcal{L} for the full subcategory of \mathcal{T}_* whose objects are the f -local spaces. Let $F : \mathcal{J} \rightarrow \mathcal{L}$ be a (small) diagram. If $\text{hocolim } L_f \circ F \in \mathcal{L}$, then the comparison map $\text{hocolim } F \rightarrow \text{hocolim } L_f \circ F$ induced by the coaugmentations is also an f -localization.*

Proposition 7 is proved in [6, Thm. 1.D.3]. It implies that if $A \xrightarrow{\alpha} B \rightarrow C$ is a cofiber sequence and $C_{L_f(\alpha)}$ is f -local, then the natural comparison map $C_\alpha \rightarrow C_{L_f(\alpha)}$ is f -localization.

The proof of the following result is entirely straightforward.

Proposition 8. *For any $f : P \rightarrow Q$ the subcategory $\mathcal{L} = \text{im}(L_f)$ is closed under weak homotopy equivalences and homotopy limits.*

In the language of [12], Proposition 8 says that \mathcal{L} is a *resolving class*.

1.2. R -Localization. Localization with respect to a ring $R \subseteq \mathbb{Q}$ is a special case of localization of considerable importance. We record its definition and basic properties here.

We write $p_n : S^n \rightarrow S^n$ for the degree p map from the n -sphere to itself (p may be any integer, but we usually take p to be a prime). Now for a subring $R \subseteq \mathbb{Q}$, let $\mathcal{P}(R)$ denote the set of all primes that are invertible in R , and define R -**localization** to be L_{q_R} where

$$q_R = \bigvee_{p \in \mathcal{P}(R)} p_1 : \bigvee_{p \in \mathcal{P}(R)} S^1 \longrightarrow \bigvee_{p \in \mathcal{P}(R)} S^1.$$

There are other (different) definitions for rationalization, but they all agree up to weak homotopy equivalence on simply-connected spaces. We write L_R for R -localization and call a q_R -equivalence an R -**equivalence**.

It is easy to detect simply-connected R -local spaces in terms of their standard algebraic invariants. An abelian group G is said to be R -**local** if the map $G \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow G \otimes_{\mathbb{Z}} R$ induced by the inclusion $\mathbb{Z} \hookrightarrow R$ is an isomorphism.

Proposition 9 (Sullivan). *Let $R \subseteq \mathbb{Q}$ be a subring. If X is simply-connected, then the following are equivalent:*

- (1) X is R -local,
- (2) $\pi_n(X)$ is R -local for all $n \geq 2$,
- (3) $\tilde{H}_n(X; \mathbb{Z})$ is R -local for all $n \geq 2$.

Together with the exactness of the R -localization of abelian groups, Proposition 9 implies the two basic homotopy-theoretical properties of R -localization of spaces.

Theorem 10 (Sullivan). *If $R \subseteq \mathbb{Q}$, then L_R commutes with both fiber sequences and cofiber sequences of simply-connected spaces.*

These results are well-known; they may be found in [13, Thm. 2.1] and [13, Thm. 2.5].

Corollary 11. *If X is $(N - 1)$ -connected, then $X \rightarrow L_{\Sigma^{N-1}q_R}(X)$ is R -localization.*

In the special case $R = \mathbb{Q}$ it is well-known that we can say quite a lot more.

Theorem 12. *For any simply-connected space X ,*

- (a) $\Sigma X_{\mathbb{Q}} \sim \bigvee_{\alpha} S_{\mathbb{Q}}^{n_{\alpha}}$ for various values of n_{α} , and
- (b) $\Omega X_{\mathbb{Q}} \sim \prod_{\alpha} K(\mathbb{Q}, m_{\beta})$ for various values of m_{β} .

This implies a closure property for resolving classes of rational spaces.

Lemma 13. *Let \mathcal{R} be a resolving class.⁴ If there is a simply-connected rational space $X \in \mathcal{R}$ with $\pi_n(X) \neq 0$, then \mathcal{R} contains all simply-connected rational spaces with $\pi_k(Y) = 0$ for $k \geq n$. If for each $n \in \mathbb{N}$ there is a rational space $X_n \in \mathcal{R}$ with $\pi_k(X_n) \neq 0$ for some $k \geq n$, then \mathcal{R} contains all simply-connected rational spaces.*

Proof. Since $K(\mathbb{Q}, n - 1)$ is a retract of $\Omega X \in \mathcal{R}$, $K(V, k) \in \mathcal{R}$ for all rational vector spaces V and all $k < n$. Now the result follows easily by induction on the Postnikov system of Y . \square

2. PROOF OF THEOREM 1

It is well-known that (1) implies (2).

Suppose we have (2). We begin our proof of (1) by studying the f -localization of $(N - 1)$ -connected spheres. Applying (2) to the cofiber sequences

$$S^n \rightarrow * \rightarrow S^{n+1} \quad \text{and} \quad S^n \xrightarrow{*} S^m \rightarrow S^m \vee S^{n+1},$$

reveals that $L_f(S^{n+1}) \simeq \Sigma L_f(S^n)$ and $L_f(S^m \vee S^{n+1}) \simeq L_f(S^m) \vee L_f(S^{n+1})$, as long as $m, n \geq N$. Inductively, we obtain weak equivalences $S^m \wedge L_f(S^n) \xrightarrow{\sim} L_f(S^m \wedge S^n)$ for all $m \geq 1$ and $n \geq N$. Next (still with $m, n \geq N$) apply L_f to the cofiber sequence

$$S^m \vee S^{n+1} \rightarrow S^m \times S^{n+1} \rightarrow S^m \wedge S^{n+1}$$

to obtain

$$\begin{aligned} L_f(S^m \wedge S^{n+1}) &\sim L_f(S^m \times S^{n+1}) / L_f(S^m \vee S^{n+1}) \\ &\sim (L_f(S^m) \times L_f(S^{n+1})) / (L_f(S^m) \vee L_f(S^{n+1})) \\ &\sim L_f(S^m) \wedge L_f(S^{n+1}), \end{aligned}$$

using Proposition 5(c).

⁴That is, \mathcal{R} is nonempty and closed under weak equivalence and homotopy limits; according to Proposition 8, $\mathcal{L} = \text{im}(L_f)$ is such a class.

We use these equivalences to construct the solid arrow part of the diagram

$$\begin{array}{ccc}
S^m \wedge S^{n+1} & \xrightarrow{\text{id}_{S^m} \wedge \iota_{S^{n+1}}} & S^m \wedge L_f(S^{n+1}) \\
\downarrow \iota_{S^m \wedge S^{n+1}} & \swarrow \iota_{S^m \wedge L_f(S^{n+1})} & \downarrow \iota_{S^m} \wedge \text{id}_{L_f(S^{n+1})} \\
L_f(S^m \wedge S^{n+1}) & \xleftarrow{\epsilon_{m,n+1}} & L_f(S^m) \wedge L_f(S^{n+1})
\end{array}$$

for $m, n \geq N$. Proposition 5(a) implies that the solid arrows in the diagram are f -equivalences. Since $L_f(S^m) \wedge L_f(S^{n+1}) \in \mathcal{L}$, the dotted arrow $\epsilon_{m,n+1}$ exists and is unique up to homotopy by the universal factorization property of L_f given in Lemma 3(b). Further, the commutativity of the diagram implies that that $\epsilon_{m,n+1}$ is also an f -equivalence. Since all the spaces in the lower right triangle are f -local, all three maps in that triangle are weak equivalences by Lemma 3(a).

We next determine the connectivity of $L_f(S^n)$. Write $\text{conn}(L_f(S^N)) = c$; since $L_f(S^N)$ is simply-connected, we have

$$\text{conn}(L_f(S^n)) = \text{conn}(\Sigma^{n-N} L_f(S^N)) = c + (n - N)$$

for all $n \geq N$, so we may work with $n > N$, and we have the equation

$$\text{conn}(L_f(S^n) \wedge L_f(S^n)) = 2(c + (n - N)) + 1.$$

Since $L_f(S^n)$ is simply-connected, the homotopy-commutative diagram of weak equivalences

$$\begin{array}{ccccc}
S^n \wedge L_f(S^n) & \xrightarrow{\sim} & L_f(S^n) \wedge L_f(S^n) & \xleftarrow{\sim} & L_f(S^n) \wedge S^n \\
& \searrow \sim & \downarrow \epsilon_{n,n} & \swarrow \sim & \\
& & L_f(S^{2n}) & &
\end{array}$$

shows that

$$n + (c + (n - N)) = \text{conn}(\Sigma^n L_f(S^n)) = \text{conn}(L_f(S^n) \wedge L_f(S^n)) = 2(c + (n - N)) + 1,$$

so $c = N - 1$ and $\text{conn}(L_f(S^n)) = n - 1$ for all $n \geq N$.

Now let $n > N$ be an even integer and write $L = L_f(S^n)$. We determine the graded abelian group $A_* = \Sigma^{-n} \tilde{H}_*(L; \mathbb{Z})$. Applying integral homology to the diagram above results in the diagram

$$\begin{array}{ccccc}
\mathbb{Z} \otimes_{\mathbb{Z}} A_* & \xrightarrow{\quad} & A_* \otimes_{\mathbb{Z}} A_* & \xleftarrow{\quad} & A_* \otimes_{\mathbb{Z}} \mathbb{Z} \\
\downarrow \cong \kappa & & \downarrow \kappa & & \downarrow \cong \kappa \\
A_* & \xrightarrow{\cong} & \Sigma^{-2n} \tilde{H}_*(L \wedge L; \mathbb{Z}) & \xleftarrow{\cong} & A_* \\
& \searrow \cong & \downarrow \cong & \swarrow \cong & \\
& & A_* & &
\end{array}$$

The commutativity of the diagram implies that the exterior product map $\kappa : A_* \otimes A_* \rightarrow \Sigma^{-2n} \tilde{H}_*(L \wedge L; \mathbb{Z})$ is an isomorphism (κ is injective by the

Künneth theorem). Write A_+ for the positive degree part of A_* . Since $(\mathbb{Z} \otimes_{\mathbb{Z}} A_+) \cap (A_+ \otimes_{\mathbb{Z}} \mathbb{Z}) = 0$ in $A_* \otimes_{\mathbb{Z}} A_*$, it must be that $A_+ \cong \mathbb{Z} \otimes_{\mathbb{Z}} A_+ = 0$ or else the vertical composite could not be injective in positive degrees. Thus A_* is some group R concentrated in degree zero, and the vertical composite reduces to an isomorphism $\mu : R \otimes_{\mathbb{Z}} R \xrightarrow{\cong} R$.

It follows directly from the commutativity and associativity of the smash product that the map μ gives R the structure of a commutative ring. Since this product is an isomorphism, R is a *solid ring* (defined and studied by Bousfield and Kan in [1]). Applying the Künneth theorem to compute A_1 (which we know to be zero), we find that $\text{Tor}_{\mathbb{Z}}(R, R) = 0$; we show that if R is a solid ring with $\text{Tor}_{\mathbb{Z}}(R, R) = 0$, then R is isomorphic to a subring of \mathbb{Q} . Let $T \subseteq R$ be the torsion subgroup. We cannot have $T = R$, for then $R \cong \mathbb{Z}/d$ by [1, Lem. 3.6], and this forces $\text{Tor}_{\mathbb{Z}}(R, R) \neq 0$. Therefore we may apply [1, Lem. 3.10], to the short exact sequence

$$0 \rightarrow T \longrightarrow R \longrightarrow R/T \rightarrow 0$$

to discover that R/T is isomorphic to a subring $S \subseteq \mathbb{Q}$ (and is therefore a flat \mathbb{Z} -module) and that T is a sum of cyclic groups. It follows that if $T \neq 0$, then $\text{Tor}_{\mathbb{Z}}(T, T) \neq 0$, and that $R \cong S \oplus T$ as abelian groups. Therefore we may compute

$$\begin{aligned} \text{Tor}_{\mathbb{Z}}(R, R) &\cong \text{Tor}_{\mathbb{Z}}(S, S) \oplus \text{Tor}_{\mathbb{Z}}(S, T) \oplus \text{Tor}_{\mathbb{Z}}(T, S) \oplus \text{Tor}_{\mathbb{Z}}(T, T) \\ &\cong \text{Tor}_{\mathbb{Z}}(T, T), \end{aligned}$$

showing that if $\text{Tor}_{\mathbb{Z}}(R, R) = 0$, then $T = 0$ and $R \cong S \subseteq \mathbb{Q}$. Thus the f -localization of S^n takes the form $\iota_{S^n} : S^n \rightarrow M(R, n)$ for all $n \geq N$.

We claim that ι_{S^n} is R -localization for all $n \geq N$. Let $\lambda : S^n \rightarrow M(R, n)$ represent $1 \in R \cong \pi_n(M(R, n))$; then $\iota_{S^n} = r \cdot \lambda$ for some $r \in R$. Since, as we showed above, L_f commutes with suspension of $(N - 1)$ -connected spheres, this r is the same for all $n \geq N$. To show that ι_{S^n} is R -localization, we need to show that r is invertible in R . Note that r cannot be zero, for if $r = 0$, then $\iota_{S^n} : S^n \rightarrow L_f(S^n)$ is nullhomotopic, forcing $L_f(S^n)$ to be contractible. Since $M(R, n)$ is f -local, $L_f(M(R, n)) \sim M(R, n)$ and the coaugmentation $M(R, n) \rightarrow L_f(M(R, n))$ can be identified with $t \cdot \text{id}$ for some invertible $t \in R$. Writing $L_f(\lambda) = s \cdot \text{id}$ for some $s \in R$, we obtain the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\lambda} & M(R, n) \\ r \cdot \lambda \downarrow & & \downarrow t \cdot \text{id} \\ M(R, n) & \xrightarrow{s \cdot \text{id}} & M(R, n). \end{array}$$

This shows that $rs = t$; and since t is invertible, so are r and s . Therefore ι_{S^n} is R -localization for $n \geq N$.⁵

⁵A similar result for smashing localizations of spectra was established in [4, Thm. 5.14].

Now we show that $\Sigma^{N-1}q_R$ is an f -equivalence; for this it suffices to show that p_N (see Section 1.2 for notation) is an f -equivalence for each $p \in \mathcal{P}(R)$. Because L_f commutes with cofiber sequences, $L_f(S^n \vee S^n) \sim L_f(S^n) \vee L_f(S^n)$ for $n > N$. Thus the co-H structure map $S^n \rightarrow S^n \vee S^n$ is carried by L_f to the co-H structure map $M(R, n) \rightarrow M(R, n) \vee M(R, n)$ for $n > N$ (both structure maps are unique up to homotopy). It follows that $L_f(\Sigma\alpha + \Sigma\beta) = \Sigma L_f(\alpha) + \Sigma L_f(\beta)$ for any $\alpha, \beta \in \pi_n(S^n)$. Since $1_n = \text{id} : S^n \rightarrow S^n$, we have $L_f(1_n) = \text{id}_{M(R, n)}$, and therefore $L_f(p_n) = p \cdot \text{id}_{M(R, n)}$. Now if p is invertible in R , then $L_f(p_n) = p \cdot \text{id}$ is a weak equivalence $M(R, n) \rightarrow M(R, n)$ for $n > N$, and also for $n = N$ by the Freudenthal suspension theorem.

Since $\Sigma^{N-1}q_R$ is an f -equivalence, there is a comparison $\xi : L_{\Sigma^{N-1}q_R} \rightarrow L_f$ which evaluates to weak equivalences ξ_{S^n} for $n \geq N$. By virtue of Proposition 6, to prove that ξ_K is a weak equivalence for all $K \in \mathcal{K}(N)$, it suffices to show that $\mathcal{K}(N) \subseteq \mathcal{L} = \text{im}(L_f)$. Since \mathcal{L} is a resolving class, the desuspension theorem [12, Thm. 8] shows that it suffices to show that the class $\Sigma\mathcal{K}(N)$ of suspensions of spaces in $\mathcal{K}(N)$ is contained in \mathcal{L} . But $\Sigma\mathcal{K}(N) \subseteq \mathcal{K}(N+1)$, so it suffices to show that $\mathcal{K}(N+1) \subseteq \mathcal{L}$; we accomplish this by induction on the number of nontrivial cells in a CW decomposition of K . The initial case is the trivial case $K = *$. For the inductive step, fit K into a cofiber sequence $S^n \xrightarrow{\alpha} L \rightarrow K$ in which $n \geq N$ and $L \in \mathcal{K}(N+1)$ has strictly fewer nontrivial cells than K . Then we may form the diagram

$$\begin{array}{ccccc} L_{\Sigma^{N-1}q_R}(S^n) & \xrightarrow{L_{\Sigma^{N-1}q_R}(\alpha)} & L_{\Sigma^{N-1}q_R}(L) & \longrightarrow & C_{L_{\Sigma^{N-1}q_R}(\alpha)} \\ \xi_{S^n} \downarrow & & \xi_L \downarrow & & \downarrow \zeta \\ L_f(S^n) & \xrightarrow{L_f(\alpha)} & L_f(L) & \longrightarrow & C_{L_f(\alpha)} \end{array}$$

in which the induced map ζ of cofibers is a weak equivalence because ξ_{S^n} and ξ_L are weak equivalences by hypothesis. Since $L_{\Sigma^{N-1}q_R}$ and L_f commute with cofiber sequences of $(N-1)$ -connected finite complexes, we have

$$L_{\Sigma^{N-1}q_R}(L) \sim C_{L_{\Sigma^{N-1}q_R}(\alpha)} \sim C_{L_f(\alpha)} \sim L_f(K).$$

This shows that $L_{\Sigma^{N-1}q_R}(K)$ is f -local, and we apply Proposition 6 to deduce that ξ_K is a weak equivalence. This establishes (1) and completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

First of all, it is well-known that \mathbb{Q} -localization satisfies the conditions of Theorem 2. Now suppose L_f satisfies those conditions. Then Theorem 1 implies that the restriction of L_f to $\mathcal{K}(2)$ is R -localization for some $R \subseteq \mathbb{Q}$. But R must be \mathbb{Q} , for if the prime p is not invertible in R , then the Moore space $M(\mathbb{Z}/p, n)$ is R -local and cannot split as copies of $L_f(S^{n_\alpha}) = M(R, n_\alpha)$, even after repeated suspension.

The proof of Theorem 1 implies that $\Sigma_{q\mathbb{Q}} : \bigvee S^2 \rightarrow \bigvee S^2$ is an f -equivalence. Since the rationalizations of all simply-connected finite complexes are in $\mathcal{L} = \text{im}(L_f)$, Lemma 13 and Corollary 11 show that the rationalization of every simply-connected space is both f -local and $\Sigma_{q\mathbb{Q}}$ -local. We deduce from Proposition 6 that $X \rightarrow L_f(X)$ is $\Sigma_{q\mathbb{Q}}$ -localization for all simply-connected spaces X ; by Corollary 11, this is rationalization.

REFERENCES

- [1] A. K. Bousfield and D. M. Kan, *The core of a ring*, J. Pure Appl. Algebra **2** (1972), 73–81. MR0308107 (46 #7222)
- [2] ———, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972. MR0365573 (51 #1825)
- [3] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94. MR0425956 (54 #13906)
- [4] Carles Casacuberta and Javier J. Gutiérrez, *Homotopical localizations of module spectra*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2753–2770 (electronic), DOI 10.1090/S0002-9947-04-03552-4. MR2139526 (2006b:55009)
- [5] Carles Casacuberta, Dirk Scevenels, and Jeffrey H. Smith, *Implications of large-cardinal principles in homotopical localization*, Adv. Math. **197** (2005), no. 1, 120–139, DOI 10.1016/j.aim.2004.10.001. MR2166179 (2006i:55013)
- [6] Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR1392221 (98f:55010)
- [7] Yves Félix and Stephen Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 1–38, DOI 10.2307/1999190. MR664027 (84h:55011)
- [8] Yves Félix and Jean-Michel Lemaire, *On the mapping theorem for Lusternik-Schnirelmann category*, Topology **24** (1985), no. 1, 41–43, DOI 10.1016/0040-9383(85)90043-6. MR790674 (86m:55005)
- [9] Joseph A. Neisendorfer, *Localization and connected covers of finite complexes*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 385–390. MR1321002 (96a:55019)
- [10] Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR0258031 (41 #2678)
- [11] Jeffrey Strom, *Miller spaces and spherical resolvability of finite complexes*, Fund. Math. **178** (2003), no. 2, 97–108, DOI 10.4064/fm178-2-1. MR2029919 (2005b:55026)
- [12] ———, *Finite-dimensional spaces in resolving classes*, Fund. Math. (to appear).
- [13] Dennis Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79. MR0442930 (56 #1305)

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